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Poincaré Polynomial of FJRW Rings and the Group-Weights Conjecture

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ABSTRACT

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FJRW-theory is a recent advancement in singularity theory arising from physics. The FJRW-theory is a graded vector space constructed from a quasihomogeneous weighted polynomial and symmetry group, but it has been conjectured that the theory only depends on the weights of the polynomial and the group. In this thesis, I prove this conjecture using Poincaré polynomials and Koszul complexes.

By constructing the Koszul complex of the state space, we have found an expression for the Poincaré polynomial of the state space for a given polynomial and associated group. This Poincaré polynomial is defined over the representation ring of a group in order for us to take G-invariants. It turns out that the construction of the Koszul complex is independent of the choice of polynomial, which proves our conjecture that two different polynomials with the same weights will have isomorphic FJRW rings as long as the associated groups are the same.

Keywords: Poincaré polynomial, FJRW theory, Group-Weights conjecture, Koszul complex

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Chapter 1. Introduction

Mirror symmetry is a phenomenon that occurs in geometry and physics. It describes relationships that show how a class of geometric objects (*A-model*) are related to a mirror dual class of objects (*B-model*). This thesis focuses on FJRW theory which provides the A-model construction of Landau-Ginzburg (LG) mirror symmetry [FJR12]. FJRW theory also contributes to the understanding of singularities which are often studied in algebraic geometry.

The A-model in LG mirror symmetry is a family of Frobenius algebras constructed from a quasihomogeneous (weighted) polynomial W and a group G of diagonal symmetries of W. Using a corresponding polynomial W^T and group G^T, we can construct the B-model, which is also a family of Frobenius algebras. The LG conjecture is that

$$\mathcal{H}_{W,G} \cong \mathcal{Q}_{W^T,G^T}$$

where $\mathcal{H}_{W,G}$ is the FJRW A-model and $\mathcal{Q}_{W^T,G^T}$ is the B-model.

Verifying the LG conjecture is challenging because it requires the understanding of the higher structures of the A and B-models. The levels of structure of these models can be summarized as first graded vector space, second Frobenius algebra and third Frobenius manifold, in order from lowest to highest. The Frobenius manifold structures are difficult to compute and in some cases are still unknown. The LG conjecture has been verified in various special cases, and at various levels in papers such as [Kra09] and [FJJS11].

The Group-Weights conjecture is a property of FJRW A-models that has long been assumed to be true but never proved. It assumes that the FJRW A-model does not depend on the choice of W, but only the weights of W and the given group.

Within the A-model it is known that both the Frobenius algebra and Frobenius manifold structures are deformation invariant and the axiom of deformation invariance tells us that the graded vector space structure, the weights of the polynomial and the symmetry group determine the entire FJRW A-model, so the Groups-Weights conjecture depends only on showing that the graded vector space is determined by the group and weights alone.

Definition 1.1. A *graded vector space* M is a direct sum $\bigoplus_{i=0}^{\infty} M_i$ where each M_i is a finite-dimensional vector space. Every element in M_i is defined to have *degree* i.

The Hilbert series is a tool used in understanding graded vector spaces. The Hilbert series (and finite equivalent, the Poincaré polynomial) is a formal power series $P(M) = \sum_{i=0}^{\infty} a_i t^i$ whose coefficients a_i are the dimensions of the corresponding M_i.

The Milnor ring $\mathcal{Q}_W$ is a key building block in the FJRW A-model. In [Arn74], we find a formula for the Poincaré polynomial of the Milnor ring $\mathcal{Q}_W$. However, since the FJRW A-model involves taking G-invariants of the Milnor ring, the usual formula for the Poincaré polynomial does not apply to the FJRW construction.

In this thesis, we attempt to find a Poincaré polynomial that would help us also keep track of the G-invariants, which was something that was not possible with the usual Poincaré polynomial formula for the Milnor ring. Representation theory is very appropriate in this setting to understand the group action on a vector space. We derive the formula for the Poincaré polynomial of the Milnor ring in terms of representations. The Group-Weights conjecture turns out to be a corollary of the formula for the Poincaré polynomial, since the formula does not depend on the actual polynomial choice but on the weights of the polynomial and choice of group.

Chapter 2. FJRW-Theory

The construction of the graded vector space structure in FJRW-theory requires a polynomial and a symmetry group.

2.1 Admissible Polynomial

The polynomial that is used has two requirements, as stated in [FJR12]: The polynomial has to be *non-degenerate* and *quasihomogeneous.*

Definition 2.1. A polynomial W is *quasihomogeneous* if there exists unique (up to scalar multiples) rational numbers $d, q_1, q_2, \ldots, q_n$ such that $W(\lambda^{q_1}x_1, \ldots, \lambda^{q_n}x_n) = \lambda^d W(x_1, \ldots, x_n)$ for all $\lambda \in \mathbb{C}$.

We will call d the total weight of the polynomial W and q_i the weight of the variable x_i.

Example 2.2. For example, let $W = x^2y + xy^4$. Since $W(\lambda^{q_x}x, \lambda^{q_y}y) = \lambda^{2q_x+q_y}x^2y + \lambda^{q_x+4q_y}xy^4$, we solve the equations

$$\begin{aligned} 2q_x + q_y &= d \\ q_x + 4q_y &= d \end{aligned}$$

and get $q_x = 3$, $q_y = 1$ and $d = 7$. So W has a total weight of 7. It is clear that the choice of q_x and q_y is unique up to scalar multiples of (q_x, q_y, d).

Definition 2.3. A quasihomogeneous polynomial W is *non-degenerate* if W contains no monomials of the form x_ix_j where $i \neq j$, and the only critical point of W is at the origin, and the degrees of W are uniquely determined up to scalar multiples.

The meaning of critical point in Definition 2.3 is *the set of points where the partial derivatives all equal zero.*

The definition of W provides us with two related objects.

Definition 2.4. The *Jabobian ideal* $\mathcal{J}$ of W is the ideal generated by the partial derivatives of W, i.e. $\mathcal{J} = \left\langle \frac{\partial W}{\partial x_1}, \dots, \frac{\partial W}{\partial x_n} \right\rangle$.

Definition 2.5. The *Milnor ring* $\mathcal{Q}_W$ of W is the quotient ring $\mathcal{Q}_W = \mathbb{C}[x_1, \dots, x_n]/\mathcal{J}$.

Theorem 2.6. *If W is a non-degenerate quasihomogeneous polynomial, then $\mathcal{Q}_W$ is finite dimensional [Arn74].*

2.2 Admissible Group

Definition 2.7. Given a polynomial, W, we define the *maximal group of diagonal symmetries* as

$$G_W^{max} = \{(\alpha_1, \dots, \alpha_n) \subset (\mathbb{C}^\times)^n | W(\alpha_1 x_1, \dots, \alpha_n x_n) = W(x_1, \dots, x_n)\}$$

Since W is quasihomogeneous, $W(\lambda^{q_1} x_1, \dots, \lambda^{q_n} x_n) = \lambda^d W(x_1, \dots, x_n)$, and hence the element $J = \left(e^{(2\pi i q_1)/d}, \dots, e^{(2\pi i q_n)/d}\right)$ is an element of G_W^{max}.

Every group used in FJRW-theory is required to be a subgroup of G_W^{max} containing $\langle J \rangle$. We call such groups "admissible".

2.3 Invertible Polynomials

A subclass of the admissible polynomials that we are interested in are invertible polynomials.

Definition 2.8. Let W be a non-degenerate quasihomogeneous polynomial with the same number of variables and monomials. Then we call W *invertible*. This name arises from the fact that if $W = \sum_{i=1}^{n} c_i \prod_{j=1}^{n} x_j^{a_{ij}}$ is non-degenerate, then W is invertible if the exponent matrix $A = (a_{ij})$ is invertible.

In [KS92], it is shown that a non-degenerate invertible polynomial can be written as a sum of invertible polynomials of the following three atomic types:

- $W_{\text{Fermat}} = x^a$ where a is a positive integer and $a \geq 2$.
- $W_{\text{loop}} = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1$ where a_i is a positive integer and $a_i \geq 2$.
- $W_{\text{chain}} = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$ where a_i is a positive integer and $a_i \geq 2$.

For most of this thesis, it is not required that the polynomials be invertible. This definition is useful in discussing different examples, especially in Chapter 3.

2.3.1 Weights of invertible polynomials. Suppose we wanted to find the weights for an invertible polynomial. We would need each monomial to have the same total weight, or in other words,

$$A \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = \begin{bmatrix} d \\ \vdots \\ d \end{bmatrix}$$

Since A is invertible, fixing d, there would be a unique solution for $q_1, \ldots, q_n$, which is rational since the entries of A are integers.

2.4 Construction of the A-model state space

We use the definition in [Kra09] for the state space of the A-Model.

Definition 2.9. Let $G \leq (\mathbb{C}^\times)^n$ and let G act on $\mathbb{C}^n$ by coordinate-wise multiplication. For each $h \in G$, we define $\text{Fix}(h) \leq \mathbb{C}^n$ to be the *fixed locus* of h.

Definition 2.10. Given a polynomial W and admissible group G, for $h \in G$, let $x_{i_1}, \ldots, x_{i_{N_h}}$ be the coordinates of $\mathbb{C}^n$ that are fixed by h. Let $\Omega^{N_h}(\text{Fix}(h))$ be the space of all top-dimensional, holomorphic differential forms on the fixed locus $\text{Fix}(h)$. We define

$$\mathcal{H}_h := \Omega^{N_h}(\text{Fix}(h))/(dW|_{\text{Fix}h} \wedge \Omega^{N_h-1}).$$

We call $\mathcal{H}_h$ the *h-sector* of W and define the *A-model state space* to be

$$\mathcal{H}_{W,G} := \bigoplus_{h \in G} (\mathcal{H}_h)^G$$

where $(\cdot)^G$ refers to all the G-invariants.

The above definition is key in proving the main result of this thesis. However, it is usually easier to compute the FJRW state space using the following equivalent definition.

Theorem 2.11. *Given a polynomial W and admissible group G,*

$$\Omega^{N_h}(\mathrm{Fix}(h))/(dW|_{Fixh} \wedge \Omega^{N_h - 1}) \cong \mathcal{Q}_{W|\,\mathrm{Fix}\,h} \cdot \omega$$

where $\mathrm{Fix}\,h$ is the fixed locus of h, N_h is the dimension of $\mathrm{Fix}\,h$ and $\mathcal{Q}_{W|\,\mathrm{Fix}\,h}$ is the Milnor ring of W restricted to $\mathrm{Fix}\,h$. Let $\omega = dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_{N_h}}$ where $x_{i_1}, \ldots, x_{i_{N_h}}$ are the coordinates of $\mathbb{C}^N$ fixed by h [Wal80a][Wal80b].

Each pair (m, h) where $h \in G$ and $m \in (\mathcal{H}_h)^G$ is called a basis element of $\mathcal{H}_{W,G}$ and is denoted by $\lceil m, h \rfloor$.

Example 2.12. Let $W = x^3y + y^5$ and $G = \left\langle \left(e^{2\pi i \frac{14}{15}}, e^{2\pi i \frac{1}{5}} \right) \right\rangle \leq \mathbb{C}_x^* \times \mathbb{C}_y^*$. Let $g = \left(e^{2\pi i \frac{14}{15}}, e^{2\pi i \frac{1}{5}} \right)$ be the generator of G.

Since $g^5 = \left(e^{2\pi i \frac{2}{3}}, 1 \right)$, we have $\mathrm{Fix}\, g^5 = \{0\} \times \mathbb{C}_y$. This gives us $W|_{\mathrm{Fix}\, g^5} = y^5$ and so $\mathcal{H}_{g^5} = \mathcal{Q}_{W|\,\mathrm{Fix}\, g^5}\, dy = \left(\mathbb{C}[y]/\langle 5y^4 \rangle \right) dy =$ $\mathrm{Span}_{\mathbb{C}}\{dy, y\, dy, y^2\, dy, y^3\, dy\}$. This is true also for g^{10}.

The element $g^{15} = 1$ is the identity $(1, 1)$ which has fixed locus $\mathrm{Fix}\, 1 = \mathbb{C}_x \times \mathbb{C}_y$. So $W|_{\mathrm{Fix}\, 1} = x^3y + y^5$ and $\mathcal{H}_1 = \mathcal{Q}_W\, dx \wedge dy$. For all the other non-trivial group elements, for example g, $W|_{\mathrm{Fix}\, g} = 0$ and so $\mathcal{H}_g = \mathbb{C} = \langle 1 \rangle$.

We can observe the action of G by looking at a basis of monomials. A basis for $\mathcal{H}_1 = \mathcal{Q}_W\, dx \wedge dy$ is $\{1\, dx \wedge dy,\ y\, dx \wedge dy,\ x\, dx \wedge dy,\ y^2\, dx \wedge dy,\ xy\, dx \wedge dy,\ x^2\, dx \wedge dy,\ y^3\, dx \wedge dy,$ $xy^2\, dx \wedge dy,\ y^4\, dx \wedge dy,\ xy^3\, dx \wedge dy,\ xy^4\, dx \wedge dy\}$. Taking the element $g = \left(e^{2\pi i \frac{14}{15}}, e^{2\pi i \frac{1}{5}} \right)$, we see for example:

$$\begin{aligned} gxy^2\,dx\wedge dy &\mapsto e^{2\pi i\frac{14}{15}}x\left(e^{2\pi i\frac{1}{5}}\right)^2 y^2\, e^{2\pi i\frac{14}{15}}dx\wedge e^{2\pi i\frac{1}{5}}dy \\ &= e^{2\pi i\frac{7}{15}}xy^2\,dx\wedge dy \end{aligned}$$

is not G-invariant. Checking each monomial, we see that $(\mathcal{H}_1)^G = \mathrm{Span}\{x^2\,dx\wedge dy\}$.

In Tables 2.1, 2.2 and 2.3, we show the computation of three sectors in this example. First we compute the monomial basis for the sectors 1, g^5 and g. The right column shows the action of g on this basis element. Since G is cyclic, if the monomial is invariant by the action of g, then the monomial is invariant by the whole group G.

From Table 2.1, $x^2\,dx\wedge dy$ is the only G-invariant monomial in $\mathcal{H}_1$. Table 2.2 tells us that from the g^5 sector, we get no G-invariants. Since $\mathrm{Fix}\,g^5 = \mathrm{Fix}\,g^{10}$, we also have no G-invariants in the g^{10} sector. For g (and equivalently g^k where $k = 2,3,4,6,7,8,9,11,12,13,14$), we get the monomial 1 which is G-invariant.

Hence for this example, a basis for $\mathcal{H}_{W,G}$ is

$$\{\lceil x^2\,dx\wedge dy, 1\rceil, \lceil 1, g\rceil, \lceil 1, g^2\rceil, \lceil 1, g^3\rceil, \lceil 1, g^4\rceil, \lceil 1, g^6\rceil, \lceil 1, g^7\rceil, \lceil 1, g^8\rceil, \lceil 1, g^9\rceil,$$
$$\lceil 1, g^{11}\rceil, \lceil 1, g^{12}\rceil, \lceil 1, g^{13}\rceil, \lceil 1, g^{14}\rceil\}.$$

Definition 2.13. Let W be a quasihomogeneous polynomial with weights $\{q_{x_i}\}$. We define the *degree* of any monomial to be the weighted sum of the corresponding q_i's.

Example 2.14. Let $W = x^2y + xy^4$ and $G = \left\langle \left(e^{2\pi i\frac{3}{7}}, e^{2\pi i\frac{1}{7}}\right)\right\rangle$. Every non-trivial $g \in G$ has nonzero entries in both coordinates, and so for each g-sector, $W|_{\mathrm{Fix}\,g} = 0$ and the g-sector $\mathcal{H}_g$ is $\mathbb{C}$.

However $g^7 = 1 = (1,1)$, so $W|_{\mathrm{Fix}\,g^7} = W$, and thus the Milnor ring is $\mathbb{C}[x,y]/\langle 2xy + y^4, x^2 + 4xy^3\rangle = \mathrm{Span}\{1, y, y^2, y^3, x, xy, xy^2, xy^3\}$, so $\mathcal{H}_1 = \mathrm{Span}\{1\,dx\wedge dy, y\,dx\wedge dy, y^2\,dx\wedge dy, y^3\,dx\wedge dy, x\,dx\wedge dy, xy\,dx\wedge dy, xy^2\,dx\wedge dy, xy^3\,dx\wedge dy\}$

$\mathcal{H}_1$	g-action
$1\,dx \wedge dy$	$e^{2\pi i \frac{2}{15}}$
$y\,dx \wedge dy$	$e^{2\pi i \frac{5}{15}}$
$x\,dx \wedge dy$	$e^{2\pi i \frac{1}{15}}$
$y^2\,dx \wedge dy$	$e^{2\pi i \frac{8}{15}}$
$xy\,dx \wedge dy$	$e^{2\pi i \frac{4}{15}}$
$x^2\,dx \wedge dy$	1 (g-invariant)
$y^3\,dx \wedge dy$	$e^{2\pi i \frac{11}{15}}$
$xy^2\,dx \wedge dy$	$e^{2\pi i \frac{7}{15}}$
$y^4\,dx \wedge dy$	$e^{2\pi i \frac{14}{15}}$
$xy^3\,dx \wedge dy$	$e^{2\pi i \frac{10}{15}}$
$xy^4\,dx \wedge dy$	$e^{2\pi i \frac{13}{15}}$

Table 2.1: Monomial basis of $\mathcal{H}_1$ where $W = x^3y + y^5$ and $1 = (1, 1)$

$\mathcal{H}_{g^5}$	g-action
$1\,dy$	$e^{2\pi i \frac{1}{5}}$
$y\,dy$	$e^{2\pi i \frac{2}{5}}$
$y^2\,dy$	$e^{2\pi i \frac{3}{5}}$
$y^3\,dy$	$e^{2\pi i \frac{4}{5}}$

Table 2.2: Monomial basis of $\mathcal{H}_{g^5}$ where $W = x^3y + y^5$ and $g^5 = (e^{2\pi i \frac{2}{3}}, 1)$

$\mathcal{H}_g$	g-action
1	1

Table 2.3: Monomial basis of $\mathcal{H}_g$ where $W = x^3y + y^5$ and $g = (e^{2\pi i \frac{14}{15}}, e^{2\pi i \frac{1}{5}})$

$\mathcal{H}_1$	Degree	g-action
$1\, dx \wedge dy$	0	$e^{2\pi i \frac{4}{7}}$
$y\, dx \wedge dy$	1	$e^{2\pi i \frac{5}{7}}$
$y^2\, dx \wedge dy$	2	$e^{2\pi i \frac{6}{7}}$
$y^3\, dx \wedge dy$	3	1
$x\, dx \wedge dy$	3	1
$xy\, dx \wedge dy$	4	$e^{2\pi i \frac{1}{7}}$
$xy^2\, dx \wedge dy$	5	$e^{2\pi i \frac{2}{7}}$
$xy^3\, dx \wedge dy$	6	$e^{2\pi i \frac{3}{7}}$

Table 2.4: Monomial basis of $\mathcal{H}_1$ of $x^2y + xy^4$ for $1 = (1, 1)$

The definition of degree as given in Definition 2.13 is the focus of this paper, however, this grading will not be preserved under the FJRW ring multiplication structure. The following grading, known as W-degree, is the actual grading for FJRW state space.

Definition 2.15. Let $h = \left(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_m}\right)$ be a group element where $0 \leq \theta_i < 1$. Then for $\alpha_h \in (H_h)^G$, the *W-degree* of α_h is $\deg_W(\alpha_h) := \dim \operatorname{Fix} h + 2\sum_{i=1}^{m}\left(\theta_i - \frac{q_i}{d}\right)$, where the q_i's are the quasihomogeneous weights of W.

Note that the W-degree of a given element is just determined by the weights q_i/d and the group element h. So to count the dimension of the space of elements of a given W-degree, it suffices to count the dimension of the space $(\mathcal{H}_h)^G$ for each $h \in G$.

Basis Element	W-Degree
$\lceil 1, \left(e^{2\pi\frac{3}{7}}, e^{2\pi\frac{1}{7}}\right)\rfloor$	0
$\lceil 1, \left(e^{2\pi\frac{2}{7}}, e^{2\pi\frac{3}{7}}\right)\rfloor$	$\frac{2}{7}$
$\lceil 1, \left(e^{2\pi\frac{1}{7}}, e^{2\pi\frac{5}{7}}\right)\rfloor$	$\frac{4}{7}$
$\lceil y^3, (1,1)\rfloor$	$\frac{6}{7}$
$\lceil x, (1,1)\rfloor$	$\frac{6}{7}$
$\lceil 1, \left(e^{2\pi\frac{6}{7}}, e^{2\pi\frac{2}{7}}\right)\rfloor$	$\frac{8}{7}$
$\lceil 1, \left(e^{2\pi\frac{5}{7}}, e^{2\pi\frac{4}{7}}\right)\rfloor$	$\frac{10}{7}$
$\lceil 1, \left(e^{2\pi\frac{4}{7}}, e^{2\pi\frac{6}{7}}\right)\rfloor$	$\frac{12}{7}$

Table 2.5: Basis for $\mathcal{H}_{W,G}$ where $W = x^2y + xy^4$ and $G = \langle e^{2\pi i\frac{3}{7}}, e^{2\pi i\frac{1}{7}}\rangle$

2.5 Construction of the FJRW ring

From the state space, the ring structure of $\mathcal{H}_{W,G}$ is determined by objects known as genus-zero three-point correlators.

Definition 2.16. Given $r, s \in \mathcal{H}_{W,G}$, we define *ring multiplication* $\star$ as $r \star s := \sum_{\alpha,\beta} \langle r, s, \alpha\rangle_{0,3}\eta^{\alpha,\beta}\beta$, where α, β are all possible elements in $\mathcal{H}_{W,G}$.

The three-point correlators, $\langle r, s, \alpha\rangle_{0,3}$ are not easy to compute and are usually computed using a set of so-called axioms.

Deformation invariance is an axiom of FJRW A-models which allow us to prove the Group-Weights conjecture without understanding the whole structure. Specifically, it says that if $G_1 = G_2$ as subgroups of $(\mathbb{C}^\times)^N$ and $\dim \mathcal{H}_{W_1,G_1} = \dim \mathcal{H}_{W_2,G_2}$, then the multiplicative structures of two FJRW rings $\mathcal{H}_{W_1,G_1}$ and $\mathcal{H}_{W_2,G_2}$ will be the same

2.6 Construction of the B-model state space

The construction of the B-model state space is similar to the construction of the A-model. However, the difference is that the admissible groups in the B-model construction do not have to contain $\langle J \rangle$, but rather have to be subgroups of $SL_n(\mathbb{C})$.

Definition 2.17. Let W be a non-degenerate quasihomogeneous polynomial. Let G be a symmetry group of W that contains $SL_n(\mathbb{C})$. For $g \in G$. Let $\text{Fix}(g) = \mathbb{C}^{N_g}$ with coordinates $x_{i_1}, \ldots, x_{i_{N_g}}$ where $N_g = \dim \text{Fix}(g)$ and let $\omega_g = dx_{i_1}, \ldots, dx_{i_{N_g}}$. We call $\mathcal{B}_g = \left(\mathcal{Q}_{W|\,\text{Fix}\,g}\right)\omega_g$ an *uprojected sector* and the *B-model state space* consists of G-invariants of these sectors:

$$\mathcal{B}_{W,G} = \bigoplus_{g \in G} (\mathcal{B}_g)^G.$$

The Group-Weights conjecture is motivated by the fact that constructing FJRW rings seems to be independent of the choice of polynomial. The formula for the Poincaré polynomial of the Milnor ring describes its graded vector space structure. This formula is independent of the actual polynomial as it only uses the quasihomogeneous weights of the polynomial. However, it is not obvious that G-invariance is independent of the polynomial since the action of G is defined to act on individual variables.

Conjecture (Group-Weights). *Let W_1 and W_2 be polynomials which have the same weights. Suppose $G \leq G_{W_1}^{max}$ and $G \leq G_{W_2}^{max}$. Then*

$$\mathcal{H}_{W_1,G} \cong \mathcal{H}_{W_2,G}$$

as Frobenius manifolds.

3.1 Common Subgroups

A first approach to proving the Group-Weights conjecture is to study common admissible subgroups of polynomials with the same weights. If W_1 and W_2 are polynomials with the same weights system, then $\langle J \rangle \leq G_{W_1}^{\max}$ and $\langle J \rangle \leq G_{W_2}^{\max}$. Recall that an admissible group in the FJRW-theory is one that contains $\langle J \rangle$. There are many examples of distinct polynomials with the same weights where the only admissible subgroup for both polynomials is $\langle J \rangle$. In fact, we can show that for two-variable invertible polynomials, $\langle J \rangle$ is the only admissible subgroup in common for distinct polynomials.

3.1.1 Common Subgroups of Two Variable Invertible Polynomials. Recall from Section 2.3, invertible quasihomogeneous polynomials are sums of polynomials of atomic types. This means that given two invertible quasihomogeneous polynomials of degree two, they would be: a sum of two Fermat type polynomials, a chain type polynomial or a loop type polynomial.

Theorem 3.1. *Let W_1 and W_2 be two-variable invertible polynomials. The only admissible subgroup of $G_{W_1}^{max} \cap G_{W_2}^{max}$ is $\langle J \rangle$.*

The proof follows from the following set of lemmas.

Lemma 3.2. *If W_{Fermat} is a two variable polynomial which is the sum of two Fermat's, i.e. $W_{Fermat} = x^m + y^n$, then the weights of W_{Fermat} are $q_x = 1/m$ and $q_y = 1/n$ with $d = 1$.*

Proof. Let $W_{Fermat} = x^m + y^n$, where $m, n \geq 2$. We can solve for the weights of each variable to get

$$q_x = \frac{1}{m}, q_y = \frac{1}{n}. \qquad \square$$

Lemma 3.3. *If W_{loop} is a two-variable polynomial that is a loop, i.e. $W_{loop} = x^a y + y^b x$, then the weights of W_{loop} are $q_x = \dfrac{b-1}{ab-1}$ and $q_y = \dfrac{a-1}{ab-1}$ with $d = 1$.*

Proof. Let W_{loop} be a two variable polynomial which is a loop. Then $W = x^a y + y^b x$ where $a, b \geq 2$. Solving for the weights of each variable, we get

$$aq_x + q_y = 1$$
$$q_x + bq_y = 1.$$

Solving this system of linear equations:

$$q_x = \frac{b-1}{ab-1}$$
$$q_y = \frac{a-1}{ab-1}. \qquad \square$$

Lemma 3.4. *If W_{loop} is a two-variable loop polynomial, i.e. $W_{loop} = x^a y + y^b x$, then the maximal symmetry group of W_{loop} is $G_{W_{loop}}^{max} = \left\langle \left(\exp\left(-2\pi i \frac{1}{ab-1}\right), \exp\left(2\pi i \frac{a}{ab-1}\right)\right)\right\rangle$. In particular, $G_{W_{loop}}^{max}$ is cyclic of order $ab-1$.*

Proof. Let $G = \left\langle \left(\exp\left(-2\pi i \frac{1}{ab-1}\right), \exp\left(2\pi i \frac{a}{ab-1}\right)\right)\right\rangle$. We see that G is a cyclic group of order $ab-1$ that fixes W_{loop}. Suppose there exists a $\left(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}\right) \in G^{max}_{W_{loop}}$. Since this element fixes the polynomial, we get

$$a\theta_1 + \theta_2 \in \mathbb{Z}$$

$$\theta_1 + b\theta_2 \in \mathbb{Z}.$$

Solving these, we get

$$(ab-1)\theta_1 \in \mathbb{Z}$$

$$(ab-1)\theta_2 \in \mathbb{Z}.$$

Hence both $e^{2\pi i\theta_1}$ and $e^{2\pi i\theta_2}$ are $(ab-1)^{\text{th}}$ roots of unity. Let $\theta_1 \equiv \dfrac{r}{ab-1} \mod \mathbb{Z}$ and $\theta_2 \equiv \dfrac{s}{ab-1} \mod \mathbb{Z}$ for $r, s \in \mathbb{Z}$. Then

$$\frac{ar+s}{ab-1} \equiv 0 \mod \mathbb{Z}$$
$$\frac{s}{ab-1} \equiv \frac{-ar}{ab-1} \mod \mathbb{Z}.$$

Hence $\left(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}\right) = \left(\exp\left(-2\pi i \frac{1}{ab-1}\right), \exp\left(2\pi i \frac{a}{ab-1}\right)\right)^{-r}$ and $G^{max}_{W_{loop}} \leq G$.

Thus $G^{max}_{W_{loop}} = \left\langle \left(\exp\left(-2\pi i \frac{1}{ab-1}\right), \exp\left(2\pi i \frac{a}{ab-1}\right)\right)\right\rangle$. □

Lemma 3.5. *If W_{chain} is a two-variable polynomial that is a forward chain, i.e $W_{chain} = x^h y + y^k$, then the weights of W_{chain} are $q_x = \dfrac{k-1}{hk}$ and $q_y = \dfrac{1}{k}$ with $d = 1$. Likewise, if W'_{chain} is a 2 variable polynomial that is a reverse chain, i.e $W'_{chain} = x^h + y^k x$, then the weights of W'_{chain} are $q_x = \dfrac{1}{h}$ and $q_y = \dfrac{h-1}{hk}$ with $d = 1$.*

Proof. Let $W_{chain} = x^h y + y^k$ be a two-variable polynomial which is a forward chain, where

$h, k \geq 2$. Solving for the weights of each variable, we get

$$\begin{aligned} hq_x + q_y &= 1 \\ hq_y &= 1 \\ q_y &= \frac{1}{h} \\ q_x &= \frac{h-1}{hk}. \end{aligned}$$

The weights for the reverse chain follows from the above calculations. $\square$

Lemma 3.6. *If W_{chain} is a two-variable chain polynomial, i.e. $W_{chain} = x^h y + y^k$, then the maximal symmetry group of W_{chain} is $G^{max}_{W_{chain}} = \left\langle \left(\exp\left(-2\pi i \frac{1}{hk}\right), \exp\left(2\pi i \frac{1}{k}\right)\right)\right\rangle$. In particular, $G^{max}_{W_{chain}}$ is cyclic of order hk.*

Proof. Let $G = \left\langle \left(\exp\left(-2\pi i \frac{1}{hk}\right), \exp\left(2\pi i \frac{1}{k}\right)\right)\right\rangle$. We see that G is a cyclic group of order hk that fixes W_{chain}. Suppose there exists a $\left(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}\right) \in G^{max}_{W_{chain}}$. Since this element fixes the polynomial, we get

$$\begin{aligned} h\theta_1 + \theta_2 \in \mathbb{Z} \\ k\theta_2 \in \mathbb{Z}. \end{aligned}$$

Solving for θ_1, we get

$$hk\theta_1 \in \mathbb{Z}.$$

Hence $e^{2\pi i\theta_1}$ is a hk^{th} root of unity and $e^{2\pi i\theta_2}$ is a k^{th} root of unity. Let $\theta_1 \equiv \frac{r}{hk} \mod \mathbb{Z}$ and $\theta_2 \equiv \frac{s}{k} \mod \mathbb{Z}$ for $r, s \in \mathbb{Z}$. Then

$$\begin{aligned} \frac{hr}{hk} + \frac{s}{k} \equiv 0 \mod \mathbb{Z} \\ \frac{s}{k} \equiv \frac{-r}{k} \mod \mathbb{Z}. \end{aligned}$$

Hence $\left(e^{2\pi i\theta_1}, e^{2\pi i\theta_2}\right) = \left(\exp\left(-2\pi i \frac{1}{hk}\right), \exp\left(2\pi i \frac{1}{k}\right)\right)^{-r}$ and $G^{max}_{W_{chain}} \leq G$.

Thus $G^{max}_{W_{chain}} = \left\langle \left(\exp\left(-2\pi i \frac{1}{hk}\right), \exp\left(2\pi i \frac{1}{k}\right) \right) \right\rangle$. $\square$

Lemma 3.7. *If W_1 and W_2 are two-variable polynomials with the same weights (q_x, q_y), and W_1 is the sum of two Fermat's $x^m + y^n$ and W_2 is a loop $x^a y + y^b x$, then $m = n$ and $a = b$. Also, $m = n = a + 1 = b + 1$.*

Proof. If W_2 had the same weights as W_1, then

$$\left(\frac{b-1}{ab-1}, \frac{a-1}{ab-1} \right) = \left(\frac{1}{m}, \frac{1}{n} \right).$$

So we get that $b-1|ab-1$ and $a-1|ab-1$.

$$\begin{aligned}
b-1|ab-1 &\implies b-1|ab-1-(b-1) \\
&\implies b-1|b(a-1) \\
&\implies b-1|a-1. \\
a-1|ab-1 &\implies a-1|ab-1-(a-1) \\
&\implies a-1|a(b-1) \\
&\implies a-1|b-1 \\
&\implies a-1 = b-1 \\
&\implies a = b.
\end{aligned}$$

And so

$$\begin{aligned}
\left(\frac{b-1}{ab-1}, \frac{a-1}{ab-1} \right) &= \left(\frac{a-1}{a^2-1}, \frac{a-1}{a^2-1} \right) \\
&= \left(\frac{1}{a+1}, \frac{1}{a+1} \right).
\end{aligned}$$

Since W_1 has the same weights vector as W_2, $m = n = a + 1 = b + 1$. $\square$

Lemma 3.8. *If W_1 and W_2 are two-variable polynomials with the same weights vector, and W_1 is the sum of two Fermat's $x^m + y^n$ and W_2 is a loop $x^a y + y^b x$, then the only admissible group in common is the group generated by J.*

Proof. By Lemma 3.7, we have weights $x = \frac{1}{a+1}$ and $q_y = \frac{1}{a+1}$. So $G_{W_1}^{max}$ is generated by $\left(\exp\left(2\pi i\frac{1}{a+1}\right), 1\right)$ and $\left(1, \exp\left(2\pi i\frac{1}{a+1}\right)\right)$, while $G_{W_2}^{max}$ is generated by $\left(\exp\left(2\pi i\frac{a}{a^2-1}\right), \exp\left(-2\pi i\frac{1}{a^2-1}\right)\right)$.

Each subgroup of $G_{W_2}^{max}$ is cyclic, and so if $G_{W_1}^{max}$ had a subgroup containing J in common with $G_{W_2}^{max}$, that subgroup would have to be cyclic. Any element of $G_{W_1}^{max}$ generates a subgroup which has order at most $a+1$, so any subgroup containing J has to have order $a+1$ and so would be $\langle J\rangle$. ◻

Lemma 3.9. *If W_1 and W_3 are two variable polynomials with the same weights vector, and W_1 is the sum of two Fermat's $x^m + y^n$ and W_3 is a chain $x^h y + y^k$, then $m = ck$ for some constant ck and $n = k$.*

Proof. If W_3 had the same weights as W_1, then

$$\left(\frac{k-1}{hk}, \frac{1}{k}\right) = \left(\frac{1}{m}, \frac{1}{n}\right)$$

So we get that $k-1 | hk$. Since $\gcd(k-1, k) = 1$, $k-1|h$, so let $\frac{h}{k-1} = c$, then

$$\left(\frac{k-1}{hk}, \frac{1}{k}\right) = \left(\frac{1}{ck}, \frac{1}{k}\right)$$

. So we see that $m = ck$ and $n = k$. ◻

Lemma 3.10. *If W_1 and W_3 are two-variable polynomials with the same weights, and W_1 is the sum of two Fermat's $x^m + y^n$ and W_3 is a chain $x^h y + y^k$, then the only admissible group in common is the group generated by J.*

Proof. By Lemma 3.9, we have that the weights would be $q_x = \frac{1}{ck}$ and $q_y = \frac{1}{k}$. The group $G_{W_1}^{max}$ is generated by $\left(\exp\left(2\pi i\frac{1}{ck}\right), 1\right)$ and $\left(1, \exp\left(2\pi i\frac{1}{k}\right)\right)$, while the group $G_{W_3}^{max}$ is generated by $\left(-\exp\left(2\pi i\frac{1}{hk}\right), \exp\left(2\pi i\frac{1}{k}\right)\right)$.

Each subgroup of $G_{W_3}^{max}$ is cyclic, and so if $G_{W_1}^{max}$ had a subgroup containing J in common with $G_{W_3}^{max}$, that subgroup would have to be cyclic. Any element of $G_{W_1}^{max}$ generates a subgroup which has order at most ck, so any subgroup containing J has to have order ck and so would be $\langle J \rangle$. $\square$

Lemma 3.11. *If W_2 and W_3 are two-variable polynomials with the same weights, and W_2 is a loop $(x^a y + y^b x)$ and W_3 is a chain $x^h y + y^k$, then $a = h$ and $q_x = \delta/k$ and $q_y = 1/k$ for some constant δ.*

Proof. If W_3 had the same weights as W_2, then $\left(\frac{b-1}{ab-1}, \frac{a-1}{ab-1}\right) = \left(\frac{k-1}{hk}, \frac{1}{k}\right)$. Solving the right coordinates:

$$\begin{aligned} \frac{a-1}{ab-1} &= \frac{1}{k} \\ \frac{ab-1}{a-1} &= k. \end{aligned} \tag{3.1.1.1}$$

Since $\frac{k-1}{hk} = \frac{b-1}{ab-1}$, substituting (3.1.1.1), we get

$$\begin{aligned} \frac{\frac{ab-1}{a-1} - 1}{h\left(\frac{ab-1}{a-1}\right)} &= \frac{b-1}{ab-1} \\ \frac{ab-1-a+1}{h(ab-1)} &= \frac{b-1}{ab-1} \\ \frac{a(b-1)}{h(ab-1)} &= \frac{b-1}{ab-1} \\ \frac{a}{h} &= 1. \end{aligned} \tag{3.1.1.2}$$

So (3.1.1.2) tells us that $a = h$ and that we get that the weights would be $\left(\frac{b-1}{hb-1}, \frac{h-1}{hb-1}\right) = \left(\frac{k-1}{hk}, \frac{1}{k}\right)$. Since $\frac{h-1}{hb-1} = \frac{1}{k}$,

$$\begin{aligned} hb - 1 &= kh - k \\ kh - hb &= k - 1 \\ h(k-b) &= k - 1. \end{aligned} \tag{3.1.1.3}$$

So (3.1.1.3) tells us that $h|k-1$, in fact $\dfrac{k-1}{h} = k-b$, so the constant δ is $k-b$. So the weights are $q_x = \dfrac{\delta}{k}$ and $q_y = \dfrac{1}{k}$. $\square$

Lemma 3.12. *If W_2 and W_3 are two-variable polynomials with the same weights, and W_2 is a loop $x^a y + y^b x$ and W_3 is a chain $x^h y + y^k$, then the only admissible group in common is the group generated by J.*

Proof. By Lemma 3.11, we have that the weights $q_x = \dfrac{\delta}{k}$ and $q_y = \dfrac{1}{k}$. The group $G^{max}_{W_2}$ is generated by $\left(\exp\left(2\pi i \dfrac{b}{hb-1}\right), \exp\left(-2\pi i \dfrac{1}{hb-1}\right)\right)$, while the group $G^{max}_{W_3}$ is generated by $\left(\exp\left(-2\pi i \dfrac{1}{hk}\right), \exp\left(2\pi i \dfrac{1}{k}\right)\right)$.

All subgroups of $G^{max}_{W_2}$ and $G^{max}_{W_3}$ are cyclic. We know from Equation (3.1.1.1) that $k|hb-1$, and $\gcd(hb-1,h) = 1$, so $\gcd(hb-1,hk) = k$. So the only subgroups that $G^{max}_{W_2}$ and $G^{max}_{W_3}$ have in common is of order k. If it is cyclic, and contains J, it has to be $\langle J \rangle$. $\square$

The preceding set of lemmas, together, prove Theorem 3.1.

Theorem 3.13. *If W_1 and W_2 have the same weights vector, then $\mathcal{H}_{W_1,\langle J\rangle} \cong \mathcal{H}_{W_2,\langle J\rangle}$ as graded vector spaces.*

This theorem was proved in [Fra12]. Given two polynomials with the same weights, the number of basis elements in the g-sector is determined by the weights. The Poincaré polynomial tells us how many monomials of the same weights there are. Hence they have to be isomorphic.

In more than two variables it is easy to find examples where the only common subgroup is $\langle J \rangle$.

Example 3.14. $W_1 = x^{13}y + xy^5 + z^4 + w^2$ and $W_2 = x^{16} + y^4 z + z^2 w + w^2$.

- $J = \left(e^{2\pi i \frac{1}{16}}, e^{2\pi i \frac{3}{16}}, e^{2\pi i \frac{1}{4}}, e^{2\pi i \frac{1}{2}}\right)$
- Common subgroup $\langle J \rangle$.

Even though it was conjectured earlier on that the only common subgroup between the symmetry groups of distinct polynomials is $\langle J\rangle$, I found examples in three or more variables that proved otherwise.

Example 3.15. $W_1 = x^7y + xy^7 + z^4 + w^2$ and $W_2 = x^7y + y^8 + z^2w + w^2$.

- $J = \left(e^{2\pi i\frac{1}{8}}, e^{2\pi i\frac{1}{8}}, e^{2\pi i\frac{1}{4}}, e^{2\pi i\frac{1}{2}}\right)$
- Common subgroup $\left\langle \left(e^{2\pi\frac{1}{8}}, e^{2\pi\frac{1}{8}}, 0, 0\right), \left(0, 0, e^{2\pi\frac{1}{4}}, e^{2\pi\frac{1}{2}}\right)\right\rangle$

In this example, we see that both W_1 and W_2 are sums of invertible polynomials in terms of x, y and z, w. So the common subgroup is not just $\langle J\rangle$ but a direct product of the respective J's of each invertible polynomial component.

Example 3.16. $X_9^T = x^3 + xy^2 + yz^2$ and $J_{10}^T = x^3 + y^3 + yz^2$.

- $J = \left(e^{2\pi i\frac{1}{3}}, e^{2\pi i\frac{1}{3}}, e^{2\pi i\frac{1}{3}}\right)$.
- Common subgroup $\left\langle \left(e^{2\pi\frac{2}{3}}, e^{2\pi\frac{2}{3}}, e^{2\pi\frac{1}{6}}\right)\right\rangle$

Here X_9^T is a chain type polynomial that is reversed while J_{10}^T is the sum of a chain type polynomial and a Fermat. The start of both chains is the common yz^2 term. This allows for the common subgroup to be slightly larger than J.

Example 3.17. Chain type polynomials can be extended to arbitrary length to produce more examples of non-$\langle J\rangle$ common subgroups.

$W_1 = x^3y + y^3z + z^4 + w^4$ and $W_2 = x^3y + y^3z + z^3w + w^4$.

- $J = \left(e^{2\pi i\frac{1}{4}}, e^{2\pi i\frac{1}{4}}, e^{2\pi i\frac{1}{4}}, e^{2\pi i\frac{1}{4}}\right)$
- Common subgroup $\left\langle \left(e^{2\pi\frac{1}{36}}, e^{2\pi\frac{11}{12}}, e^{2\pi\frac{1}{4}}, e^{2\pi\frac{1}{4}}\right)\right\rangle$

$W_1 = x_1^3x_2 + x_2^3x_3 + x_3^3x_4 + x_4^3x_5 + x_5^4$ and $W_2 = x_1^3x_2 + x_2^3x_3 + x_3^4 + x_4^4 + x_5^4$.

- $J = \left(e^{2\pi i\frac{1}{4}}, e^{2\pi i\frac{1}{4}}, e^{2\pi i\frac{1}{4}}, e^{2\pi i\frac{1}{4}}, e^{2\pi i\frac{1}{4}}\right)$
- Common subgroup $\left\langle \left(e^{2\pi\frac{1}{36}}, e^{2\pi\frac{11}{12}}, e^{2\pi\frac{1}{4}}, e^{2\pi\frac{1}{4}}, e^{2\pi\frac{1}{4}}\right)\right\rangle$

3.2 FJRW Ring Multiplication and Deformation Invariance

Even for polynomials W_1 and W_2 where the common subgroup of $G_{W_1}^{max}$ and $G_{W_2}^{max}$ is known, finding an isomorphism between their respective FJRW rings is difficult. Recall from Section 2.5 that multiplication in FJRW rings require the hard task of finding so-called three-point correlators. We will not give details about theses correlators here, but we describe several examples with their multiplicative structures.

Example 3.18. Let $V = x^9 + y^3$ and let $W = x^6y + y^3$. Then both V and W a non-degenerate polynomials with quasihomogeneous weights $q_x = 1$ and $q_y = 3$. Let G be the group $G = \langle J \rangle = \langle e^{2\pi i \frac{1}{9}}, e^{2\pi i \frac{1}{3}} \rangle$.

Name	Basis Element	W-Degree
v_1	$\left\lceil 1, \left(e^{2\pi \frac{1}{9}}, e^{2\pi \frac{1}{3}}\right) \right\rfloor$	0
v_2	$\left\lceil 1, \left(e^{2\pi \frac{4}{9}}, e^{2\pi \frac{1}{3}}\right) \right\rfloor$	2/3
v_3	$\left\lceil 1, \left(e^{2\pi \frac{2}{9}}, e^{2\pi \frac{2}{3}}\right) \right\rfloor$	8/9
v_4	$\lceil x^2y, (1,1) \rfloor$	10/9
v_5	$\lceil x^5, (1,1) \rfloor$	10/9
v_6	$\left\lceil 1, \left(e^{2\pi \frac{7}{9}}, e^{2\pi \frac{1}{3}}\right) \right\rfloor$	4/3
v_7	$\left\lceil 1, \left(e^{2\pi \frac{5}{9}}, e^{2\pi \frac{2}{3}}\right) \right\rfloor$	14/9
v_8	$\left\lceil 1, \left(e^{2\pi \frac{8}{9}}, e^{2\pi \frac{2}{3}}\right) \right\rfloor$	20/9

Table 3.1: Basis for $\mathcal{H}_{V,G}$ where $\mathcal{H}_{V,G}$ where $V = x^9 + y^3$ and $G = \langle e^{2\pi i \frac{1}{9}}, e^{2\pi i \frac{1}{3}} \rangle$

Name	Basis Element	W-Degree
w_1	$\lceil 1, \left(e^{2\pi\frac{1}{9}}, e^{2\pi\frac{1}{3}}\right) \rfloor$	0
w_2	$\lceil 1, \left(e^{2\pi\frac{4}{9}}, e^{2\pi\frac{1}{3}}\right) \rfloor$	$2/3$
w_3	$\lceil 1, \left(e^{2\pi\frac{2}{9}}, e^{2\pi\frac{2}{3}}\right) \rfloor$	$8/9$
w_4	$\lceil x^2y, (1,1) \rfloor$	$10/9$
w_5	$\lceil x^5, (1,1) \rfloor$	$10/9$
w_6	$\lceil 1, \left(e^{2\pi\frac{7}{9}}, e^{2\pi\frac{1}{3}}\right) \rfloor$	$4/3$
w_7	$\lceil 1, \left(e^{2\pi\frac{5}{9}}, e^{2\pi\frac{2}{3}}\right) \rfloor$	$14/9$
w_8	$\lceil 1, \left(e^{2\pi\frac{8}{9}}, e^{2\pi\frac{2}{3}}\right) \rfloor$	$20/9$

Table 3.2: Basis for $\mathcal{H}_{W,G}$ where $\mathcal{H}_{W,G}$ where $W = x^6y + y^3$ and $G = \langle e^{2\pi i \frac{1}{9}}, e^{2\pi i \frac{1}{3}} \rangle$

$\cdot$	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
W-Degree	0	$2/3$	$8/9$	$10/9$	$10/9$	$4/3$	$14/9$	$20/9$
v_1	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
v_2	v_2	v_6	v_7	0	0	0	v_8	0
v_3	v_3	v_7	0	0	0	v_8	0	0
v_4	v_4	0	0	0	$\frac{1}{27}v_8$	0	0	0
v_5	v_5	0	0	$\frac{1}{27}v_8$	0	0	0	0
v_6	v_6	0	v_8	0	0	0	0	0
v_7	v_7	v_8	0	0	0	0	0	0
v_8	v_8	0	0	0	0	0	0	0

Table 3.3: Multiplication Table for $\mathcal{H}_{V,G}$ where $V = x^9 + y^3$ and $G = \langle e^{2\pi i \frac{1}{9}}, e^{2\pi i \frac{1}{3}} \rangle$

$\cdot$	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
W-Degree	0	2/3	8/9	10/9	10/9	4/3	14/9	20/9
w_1	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8
w_2	w_2	w_6	w_7	0	0	0	w_8	0
w_3	w_3	w_7	0	0	0	w_8	0	0
w_4	w_4	0	0	$\frac{1}{18}w_8$	0	0	0	0
w_5	w_5	0	0	0	$-\frac{1}{6}w_8$	0	0	0
w_6	w_6	0	w_8	0	0	0	0	0
w_7	w_7	w_8	0	0	0	0	0	0
w_8	w_8	0	0	0	0	0	0	0

Table 3.4: Multiplication Table for $\mathcal{H}_{W,G}$ where $W = x^6y + y^3$ and $G = \langle e^{2\pi i \frac{1}{9}}, e^{2\pi i \frac{1}{3}} \rangle$

In this example, we see that for the state space of both $\mathcal{H}_{V,G}$ and $\mathcal{H}_{W,G}$ are of the same dimension in each degree. Comparing their multiplication tables, there is an obvious map between most of the elements in the ring. However, the problem arises for the elements with W-degree $\frac{10}{9}$.

$\cdot$	v_4	v_5
v_4	0	$\frac{1}{27}v_8$
v_5	$\frac{1}{27}v_8$	0

Table 3.5: Multiplication Table for $\mathcal{H}_{V,G}$ for elements of W-degree $\frac{10}{9}$

$\cdot$	w_4	w_5
w_4	$\frac{1}{18}w_8$	0
w_5	0	$-\frac{1}{6}w_8$

Table 3.6: Multiplication Table for $\mathcal{H}_{W,G}$ for elements of W-degree $\frac{10}{9}$

An isomorphism of these spaces as Frobenius algebras will have to send v_4 and v_5 each to some linear combination of w_4 and w_5.

With more variables and higher degree polynomials, the examples become increasingly more complicated.

3.2.1 Deformation Invariance. One of the properties of the FJRW rings is deformation invariance. *Deformation invariance* tells us that the Frobenius manifold structure of the FJRW-theory (otherwise known as virtual cycle) is dependent only on the associated state space. In other words, if $\mathcal{H}_{W_1,G_1} \cong \mathcal{H}_{W_2,G_2}$ as graded vector spaces, then $\mathcal{H}_{W_1,G_1} \cong \mathcal{H}_{W_2,G_2}$ in the FJRW-theory. This allows us to avoid actually having to find a ring isomorphism.

In Chapter 6, I will prove the conjecture by proving that the graded vector space of a FJRW ring is defined only by the weights of W and the symmetry group G. This will be a corollary of the computation of the Poincaré polynomial for $\mathcal{H}_{W,G}$.

4.1 Representations

A group representation is a way of visualizing a group G as a group of linear transformation of n-dimensional vector space. For the purposes of this thesis, we shall only consider $\mathbb{C}$-vector spaces.

Definition 4.1. A *representation* of G over $\mathbb{C}$ is a homomorphism ρ from G to $GL(n, \mathbb{C})$, for some n. We call n the *degree* of the representation ρ.

Another way of understanding representations is to think of $\mathbb{C}G$-modules, which are vector spaces that have a G action on them.

Definition 4.2. Let V be a vector space over $\mathbb{C}$ and let G be a group. V is a $\mathbb{C}G$-*module* if the action gv for $v \in V$ and $g \in G$ satisfies the following conditions for all $u, v \in V$, $\lambda \in \mathbb{C}$ and $g, h \in G$.

(i) $gv \in V$;

(ii) $(gh)v = g(hv)$;

(iii) $1v = v$;

(iv) $\lambda(gv) = g(\lambda v)$;

(v) $g(u + v) = gu + gv$.

Definition 4.3. A $\mathbb{C}G$-*submodule* of V is a vector subspace of V which is a $\mathbb{C}G$-module.

Example 4.4. Consider the cyclic group $C_3 = \langle 1, g, g^2 \rangle$ [JL01].

Let $V = \text{Span}_{\mathbb{C}}\{v_1, v_2\}$, where g acts by $\rho(g) = \begin{bmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{4\pi i/3} \end{bmatrix}$ and g^2 acts by $\rho(g^2) = \rho(g)^2 = \begin{bmatrix} e^{4\pi i/3} & 0 \\ 0 & e^{2\pi i/3} \end{bmatrix}$. Then V is a $\mathbb{C}C_3$-module.

Since $gv_1 = e^{2\pi i \frac{1}{3}} v_1$ and $gv_2 = e^{2\pi i \frac{2}{3}} v_2$, and $g^2 v_1 = e^{2\pi i \frac{2}{3}} v_1$ and $g^2 v_2 = e^{2\pi i \frac{1}{3}} v_2$, we get two $\mathbb{C}C_3$-submodules of V by Definition 4.3, namely $V_1 = \text{Span}_{\mathbb{C}}\{v_1\}$ and $V_2 = \text{Span}_{\mathbb{C}}\{v_2\}$.

Now we consider the group ring $\mathbb{C}C_3 = \mathrm{Span}_{\mathbb{C}}\{1, g, g^2\}$. The group ring is also a $\mathbb{C}C_3$-module. From the group ring, we get a corresponding representation ρ. Since $g \cdot 1 = g$, $g \cdot g = g^2$ and $g \cdot g^2 = 1$, we get that $\rho(g) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

The group ring $\mathbb{C}C_3$ has three $\mathbb{C}C_3$-submodules:

$U_0 = \mathrm{Span}_{\mathbb{C}}\{1 + g + g^2\}$,

$U_1 = \mathrm{Span}_{\mathbb{C}}\{1 + e^{2\pi i \frac{2}{3}} g + e^{2\pi i \frac{1}{3}} g^2\}$ and

$U_2 = \mathrm{Span}_{\mathbb{C}}\{1 + e^{2\pi i \frac{1}{3}} g + e^{2\pi i \frac{2}{3}} g^2\}$. If we take the element $1 + e^{2\pi i \frac{2}{3}} g + e^{2\pi i \frac{1}{3}} g^2 \in U_1$ and act on it by g, we get $g + e^{2\pi i \frac{2}{3}} g^2 + e^{2\pi i \frac{1}{3}} = e^{2\pi i \frac{1}{3}}(1 + e^{2\pi i \frac{2}{3}} g + e^{2\pi i \frac{1}{3}} g^2)$. Hence we see that U_1 is isomorphic to V_1 in the earlier example.

Definition 4.5. A representation is *irreducible* if the corresponding $\mathbb{C}G$-module has no proper non-trivial $\mathbb{C}G$-submodules.

The following are three classical theorems in representation theory.

Theorem 4.6. *Every irreducible $\mathbb{C}G$-module is isomorphic to a $\mathbb{C}G$-submodule of the group ring $\mathbb{C}G$.*

Theorem 4.7. *Let G be an abelian group. Every irreducible $\mathbb{C}G$-module is one dimensional.*

This property is useful for us since G_W^{max} is abelian.

Definition 4.8. Let ρ be a representation of G. The *character* of ρ is a function $\chi : G \to \mathbb{C}$, where $\chi(g) = \operatorname{tr} \rho(g)$ for each $g \in G$.

Theorem 4.9. *Let ρ be a representation of G. The dimension of the $\mathbb{C}G$-module corresponding to ρ is* $\operatorname{tr} \rho(1_G) = \chi(1_G)$.

4.2 Representation Ring

Definition 4.10. Given representations $\rho_1 : G \to GL(V_1)$ and $\rho_2 : G \to GL(V_2)$ of G, we define $\rho_1 + \rho_2$ and $\rho_1 \times \rho_2$ to be homomorphisms

$$\begin{aligned} \rho_1 + \rho_2 &: G \to GL(V_1 \oplus V_2) \\ \rho_1 \times \rho_2 &: G \to GL(V_1 \otimes V_2) \end{aligned}$$

where $(\rho_1 + \rho_2)(g) = \rho_1(g) \oplus \rho_2(g)$ and $(\rho_1 \times \rho_2)(g) = \rho_1(g) \otimes \rho_2(g)$

Definition 4.11. The *representation ring* $R(G)$ is the ring generated by all representations of G over $\mathbb{C}$ using the operations defined in Definition 4.10 and a formal additive inverse defined in the obvious way.

By Theorem 4.6, $R(G)$ is generated by all irreducible representations of G.

Example 4.12. Once again, we let $G = C_3$. Let V_1 and V_2 be the $\mathbb{C}C_3$-submodules of V as in Example 4.3. Let ρ_1 and ρ_2 be representations corresponding to V_1 and V_2 respectively.

$\rho_1(g) = \begin{bmatrix} e^{2\pi i \frac{1}{3}} \end{bmatrix}$ and $\rho_2(g) = \begin{bmatrix} e^{2\pi i \frac{2}{3}} \end{bmatrix}$. So $\rho_1 + \rho_2(g) = \begin{bmatrix} e^{2\pi i \frac{1}{3}} & 0 \\ 0 & e^{2\pi i \frac{2}{3}} \end{bmatrix}$.

$\rho_1 \times \rho_2(g)(v_1 \otimes v_2) = e^{2\pi i \frac{1}{3}} v_1 \otimes e^{2\pi i \frac{2}{3}} v_2 = v_1 \otimes v_2$. So $\rho_1 \times \rho_2$ is the trivial representation $1_{R(C_3)}$.

The representation ring $R(C_3)$ is the formal ring generated by $1, \rho_1, \rho_2$. Since $\rho_2 = \rho_1^2$, we get that $R(C_3)$ is generated as a ring by ρ_1 and satisfies $\rho_1^3 = 1$. Hence $R(C_3)$ is isomorphic to the group ring $\mathbb{Z}C_3$.

Recall from the Introduction the following definition and theorem.

Definition 5.1. The *Hilbert series* of a graded vector space M is a formal power series

$$P(M) = \sum_{i=0}^{\infty} a_i t^i,$$

where a_i is the dimension of M_i.

When $P(M)$ is finite, we call $P(M)$ the *Poincaré polynomial*.

Theorem 5.2. *Let W be a quasihomogeneous polynomial with weights $d, q_1, q_2, \ldots, q_n$. The Hilbert series of the Milnor ring $\mathcal{Q}_W$ is*

$$P(\mathcal{Q}_W) = \prod_{i=1}^{n} \frac{1 - t^{d-q_i}}{1 - t^{q_i}}.$$

Note that according to Theorem 2.6, $P(\mathcal{Q}_W)$ is a polynomial when W is non-degenerate. We will prove a more refined version of Theorem 5.2 in Chapter 6.

Since the Poincaré polynomial tells us the dimension of each subspace of a certain degree, if we let $t = 1$, we would get the dimension of the entire space.

Theorem 5.3. *The dimension of $\mathcal{Q}_W$ is $\mu = \prod_{i=1}^{n} \left(\frac{d}{q_i} - 1 \right)$.*

Proof. For each term in the product of $P(\mathcal{Q}_W)$, we get that 1 is a root of both the numerator and the denominator.

$$\begin{aligned} \frac{1 - t^{d-q_i}}{1 - t^{q_i}} &= \frac{\cancel{(1-t)}(1 + t + t^2 + \ldots + t^{d-q_i-1})}{\cancel{(1-t)}(1 + t + t^2 + \ldots + t^{q_i-1})} \\ &= \frac{(1 + t + t^2 + \ldots + t^{d-q_i-1})}{(1 + t + t^2 + \ldots + t^{q_i-1})} \end{aligned}$$

letting $t = 1$,

$$\begin{aligned} &= \frac{d - q_i}{q_i} \\ &= \frac{d}{q_i} - 1. \end{aligned}$$

□

5.1 Representation-Valued Poincaré Polynomial

Definition 5.4. Let $M = \bigoplus_{i=0}^{\infty} M_i$ be a graded $\mathbb{C}$ vector space. Let G be a group which acts on M such that M is a $\mathbb{C}G$-module and each M_i is a $\mathbb{C}G$-submodule with corresponding representation ρ_i (where ρ_i is not necessarily irreducible). We define the *representation-valued Hilbert series of M* to be

$$P(M, R(G) = \sum_{i=1}^{\infty} \rho_i t^i$$

where $R(G)$ is the representation ring of G.

Note that by taking the trace (character) of this expression at $1 \in G$, we get the usual Hilbert series.

This definition relies on the group G preserving the grading of M, i.e. $gm_i \in M_i$ for all $m_i \in M_i$ and $g \in G$. The action of the admissible groups on our state space multiplies a basis element by a complex root of unity as shown in Example 2.12 and thus preserves the grading. Hence, we can define the Hilbert series with representation for a FJRW state space. In fact, since the FJRW state space is finite dimensional, we can call it a *representation-valued Poincaré polynomial.*

5.2 Poincaré Polynomial on Exact Sequences

Definition 5.5. Let K be a sequence of vector spaces and linear transformations:

$$0 \xrightarrow{d_0} N_1 \xrightarrow{d_1} N_2 \xrightarrow{d_2} \ldots \longrightarrow N_{n-1} \xrightarrow{d_{n-1}} N_n \xrightarrow{d_n} \ldots$$

the sequence is *exact* at N_i if $\ker d_i = \operatorname{Im} d_{i-1}$ and K is *exact* if it is exact at all N_i.

Definition 5.6. Let $d : M \to N$ be a linear transformation where M and N are graded vector spaces. We say that d has *degree* b if $d(m_i)$ has degree $i + b$ in N for every m_i of degree i in M.

If $b = 0$, we say that d is *degree-preserving.*

Proposition 5.7. *Let K be an exact sequence of vector spaces:*

$$0 \xrightarrow{d_0} N_1 \xrightarrow{d_1} N_2 \xrightarrow{d_2} \ldots \longrightarrow N_{n-1} \xrightarrow{d_{n-1}} N_n \xrightarrow{d_n} 0$$

where each d_i has degree b_i, then $\sum_{i=1}^{n}(-1)^i t^{\beta_i} P(N_i) = 0$ *where* $\beta_i = \sum_{j=i}^{n-1} b_j$.

Proof. For each vector space N_i, we have $N_i \cong \ker d_i \oplus \operatorname{Im} d_i$. Suppose n is odd, then

$$\begin{aligned}
& N_1 \oplus N_3 \oplus \ldots \oplus N_n \\
\cong\ & \ker d_1 \oplus \operatorname{Im} d_1 \oplus \ker d_3 \oplus \operatorname{Im} d_3 \oplus \ker d_5 \oplus \ldots \oplus \operatorname{Im} d_{n-2} \oplus \ker d_n \oplus \operatorname{Im} d_n \\
\cong\ & \operatorname{Im} d_1 \oplus \ker d_3 \oplus \operatorname{Im} d_3 \oplus \ker d_5 \oplus \ldots \oplus \operatorname{Im} d_{n-2} \oplus \ker d_n,
\end{aligned}$$

since $\ker d_1 = \operatorname{Im} d_0 = 0$ and $\operatorname{Im} d_n = 0$

$$\cong\ \ker d_2 \oplus \operatorname{Im} d_2 \oplus \ker d_4 \oplus \operatorname{Im} d_4 \oplus \ldots \oplus \ker d_{n-1} \oplus \operatorname{Im} d_{n-1},$$

by the condition of exactness in Definition 5.5

$$\cong\ N_2 \oplus N_4 \oplus \cdots \oplus N_{n-1}.$$

Similarly, if n is even, then

$$\begin{aligned}
& N_1 \oplus N_3 \oplus \ldots \oplus N_{n-1} \\
\cong\ & \ker d_1 \oplus \operatorname{Im} d_1 \oplus \ker d_3 \oplus \operatorname{Im} d_3 \oplus \ker d_5 \oplus \ldots \oplus \operatorname{Im} d_{n-3} \oplus \ker d_{n-1} \oplus \operatorname{Im} d_{n-1} \\
\cong\ & \operatorname{Im} d_1 \oplus \ker d_3 \oplus \operatorname{Im} d_3 \oplus \ker d_5 \oplus \ldots \oplus \operatorname{Im} d_{n-3} \oplus \ker d_{n-1} \oplus \operatorname{Im} d_{n-1},
\end{aligned}$$

since $\ker d_1 = \operatorname{Im} d_0 = 0$

$$\cong\ \ker d_2 \oplus \operatorname{Im} d_2 \oplus \ker d_4 \oplus \operatorname{Im} d_4 \oplus \ldots \oplus \ker d_{n-2} \oplus \operatorname{Im} d_{n-2} \oplus \ker d_n \oplus \operatorname{Im} d_n,$$

by the condition of exactness in Definition 5.5 and since $\operatorname{Im} d_n = 0$

$$\cong\ N_2 \oplus N_4 \oplus \cdots \oplus N_n.$$

We now use the following diagram to illustrate the logic for the rest of the proof.

$$\begin{tikzcd}
 & \ker d_{n-2} \arrow[r, dash] \arrow[d, dash] & \operatorname{Im} d_{n-2} \arrow[d, dash] & \ker d_n \arrow[d, dash] \\
\dots \arrow[r, dash] & \operatorname{Im} d_{n-3} & \ker d_{n-1} \arrow[r, dash] & \operatorname{Im} d_{n-1}
\end{tikzcd}$$

We start from $\ker d_n = N_n$. From here, we know that $P(N_n) = P(\ker d_n)$. Using $N_{n-1} = \ker d_{n-1} \oplus d_{n-1}^{-1}(\operatorname{Im} d_{n-1})$, we get $P(N_{n-1}) = P(\ker d_{n-1}) + P(d_{n-1}^{-1}(\operatorname{Im} d_{n-1}))$. To match the terms from N_{n-1} with the terms from N_n, we multiply by $t^{b_{n-1}}$ and get $t^{b_{n-1}}P(N_{n-1}) = t^{b_{n-1}}P(\ker d_{n-1}) + t^{b_{n-1}}P(d_{n-1}^{-1}(\operatorname{Im} d_{n-1})) = t^{b_{n-1}}P(\ker d_{n-1}) + P(\operatorname{Im} d_{n-1})$.

Next, from $P(N_{n-2}) = P(\ker d_{n-2}) + P(d_{n-2}^{-1}(\operatorname{Im} d_{n-2}))$, in order to match it with $P(N_n)$, we multiply by $t^{b_{n-2}}t^{b_{n-1}}$ since $t^{b_{n-2}}t^{b_{n-1}}P(d_{n-2}^{-1}(\operatorname{Im} d_{n-2})) = t^{b_{n-1}}P(\operatorname{Im} d_{n-2}) = t^{b_{n-1}}P(\ker d_{n-1})$.

So in order to match all of the terms, we have to multiply each $P(N_i)$ by t^{β_i} where β_i is the sum of all the degrees of d_i proceeding N_i up to d_{n-1}. $\square$

Corollary 5.8. *Let K be an exact sequence of graded vector spaces:*

$$0 \xrightarrow{d_0} N_1 \xrightarrow{d_1} N_2 \xrightarrow{d_2} \dots \longrightarrow N_{n-1} \xrightarrow{d_{n-1}} N_n \xrightarrow{d_n} 0$$

where each d_i is a degree-preserving map, then $\sum_{i=1}^{n}(-1)^i P(N_i) = 0$.

Definition 5.9. Let g have an action on sets M and N. We say that the map $\varphi : M \to N$ is *equivariant* if for $m \in M$, $g\varphi(m) = \varphi(gm)$.

Proposition 5.10. *Let K be an exact sequence of graded vector spaces, with G acting on each N_i in a way that is degree-preserving:*

$$0 \xrightarrow{d_0} N_1 \xrightarrow{d_1} N_2 \xrightarrow{d_2} \dots \longrightarrow N_{n-1} \xrightarrow{d_{n-1}} N_n \xrightarrow{d_n} 0\,.$$

If each d_i has degree b_i and is equivariant, then $\sum_{i=1}^{n}(-1)^i t^{\beta_i} P(N_i, R(G)) = 0$ *where* $\beta_i =$

$\sum_{j=i}^{n-1} b_j.$

Proof. We follow from the proof of Proposition 5.7.

$$
\begin{tikzcd}
 & \ker d_{n-2} \arrow[r, dash] \arrow[d, dash] & \operatorname{Im} d_{n-2} \arrow[d, dash] & \ker d_n \arrow[d, dash] \\
\ldots \arrow[r, dash] & \operatorname{Im} d_{n-3} & \ker d_{n-1} \arrow[r, dash] & \operatorname{Im} d_{n-1}
\end{tikzcd}
$$

Once again we start from $\ker d_n = N_n$. Since $N_{n-1} = \ker d_{n-1} \oplus d_{n-1}^{-1}(\operatorname{Im} d_{n-1})$ and d_{n-1} is equivariant, both $\ker d_{n-1}$ and $d_{n-1}^{-1}(\operatorname{Im} d_{n-1})$ are $\mathbb{C}G$-submodules of N_{n-1}. The representation of $d_{n-1}^{-1}(\operatorname{Im} d_{n-1})$ is also the same as the representation of $\ker d_n$. Once again, in order to match the terms from N_{n-1} with the terms from N_n, we multiply by $t^{b_{n-1}}$ and get $t^{b_{n-1}}P(N_{n-1}, R_G(\mathbb{C})) = t^{b_{n-1}}P(\ker d_{n-1}) + t^{b_{n-1}}P(d_{n-1}^{-1}(\operatorname{Im} d_{n-1}), , R_G(\mathbb{C})) = t^{b_{n-1}}P(\ker d_{n-1}) + P(\operatorname{Im} d_{n-1}, R_G(\mathbb{C}))$.

Just as in the proof of Proposition 5.7, we continue matching terms, and since at each stage, the map d_i is equivariant, the representations all remain the same. $\square$

A useful tool in computing the Hilbert series of the FJRW state space is the Koszul complex. We use the fact that the partial derivatives of a non-degenerate polynomial forms a regular sequence. From this regular sequence, the Koszul complex gives us a long exact sequence where the final term in the sequence is the Milnor Ring.

6.1 Regular Sequence

Definition 6.1. Let R be a ring and let M be an R-module. A sequence of elements $r_1, \ldots, r_n \in R$ is called a *regular sequence* on M if $(r_1, \ldots, r_n)M \neq M$, and for $i = 1, \ldots, n$, r_i is a nonzerodivisor of $M/(r_1, \ldots, r_{i-1})M$.

Using the method given in [Stu93], we introduce the following terminology.

Definition 6.2. Let R be an associative and commutative graded k-algebra, where k is a field. A sequence of elements $r_1, \ldots, r_n \in R$ is called a *homogeneous system of parameters (h.s.o.p.)* of R if $R/\langle r_1, \ldots, r_n \rangle$ is a finite dimensional k-vector space.

Theorem 6.3. *Let R be a Noetherian graded k-algebra. If R has a h.s.o.p. that is a regular sequence, then any h.s.o.p. in R is a regular sequence.*

The proof of this theorem can be found in [Stu93] and [Sta79], and is further explained in [Pic06].

Proposition 6.4. *Let W be a nondegenerate polynomial in $\mathbb{C}[x_1, \ldots, x_n]$. Then the set of partial derivatives $\{W_{x_i}\}$ is a regular sequence on $\mathbb{C}[x_1, \ldots, x_n]$.*

Proof. $x_1, \ldots, x_n$ is a h.s.o.p. and is also a regular sequence. Recall from Theorem 2.6 that the nondegeneracy of W tells us that $\mathbb{C}[x_1, \ldots, x_n]/\langle W_{x_1}, \ldots, W_{x_n} \rangle$ is a finite dimensional $\mathbb{C}$ vector space and thus $W_{x_1}, \ldots, W_{x_n}$ is a h.s.o.p. Since $\mathbb{C}[x_1, \ldots, x_n]$ is Noetherian, by Theorem 6.3, $W_{x_1}, \ldots, W_{x_n}$ is a regular sequence. $\square$

6.2 Koszul Complex

The Koszul complex that we are going to define is a sequence of graded vector spaces and this section gives us some definitions to help us understand its construction and properties.

Definition 6.5. Let R be a commutative ring with unity. A sequence K of R-modules:

$$0 \xrightarrow{d_0} N_0 \xrightarrow{d_1} N_1 \xrightarrow{d_2} \dots \longrightarrow N_{n-1} \xrightarrow{d_n} N_n \xrightarrow{d_{n+1}} \dots$$

where each d_i is a module homomorphism and $d_{n+1} \circ d_n = 0$ for all n is called a *complex*. The n^{th} *cohomology group* is defined to be

$$H^n(K) = \ker d_{n+1} / \operatorname{Im} d_n.$$

To illustrate how the Koszul complex is constructed, we start with the Koszul complex in 1 and 2 variables. We denote the Koszul complex in one variable as $K(W_x\, dx)$ and it is the sequence below:

$$0 \longrightarrow R \xrightarrow{r_x} R\, dx \longrightarrow 0,$$

where $r_x(f) = f \wedge W_x\, dx$ for $f \in R = \mathbb{C}[x]$.

Certainly r_x is injective since W_x is a nonzerodivisor (i.e. $H^0(K(W_x dx)) = 0$). Now we compute $H^1(K(W_x\, dx)) = R\, dx / W_x R\, dx = R/(W_x)\, dx$. The Koszul complex can be extended to an exact sequence

$$0 \longrightarrow R \xrightarrow{r_x} R\, dx \longrightarrow R/(W_x)\, dx \longrightarrow 0$$

Now let $R = \mathbb{C}[x, y]$. The Koszul complex in 2 variables $K(W_x\, dx, W_y\, dy)$ is motivated by the following complex

$$
\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "r_x"] \arrow[dr, "r_y"] & R\,dx \arrow[r] \arrow[dr, "r_y"] & 0 & \\
 & 0 \arrow[r] & R\,dy \arrow[r, "-r_x"] & R\,dx \wedge dy \arrow[r] & 0
\end{tikzcd}
$$

where $r_x(f) = f \wedge W_x\,dx$ and $r_y(f) = f \wedge W_y\,dy$.

$K(W_x\,dx, W_y\,dy)$ is the following complex

$$
0 \longrightarrow R \xrightarrow{s_1} R\,dx \oplus R\,dy \xrightarrow{s_2} R\,dx \wedge dy \longrightarrow 0
$$

where both s_1, s_2 are given by $r \mapsto r \wedge dW$. That is

$$
\begin{aligned}
s_1(r) &= r \wedge dW \\
&= r \wedge (W_x\,dx \oplus W_y\,dy) \\
&= (r \wedge W_x\,dx, r \wedge W_y\,dy)
\end{aligned}
$$

And,

$$
\begin{aligned}
s_2(a\,dx, b\,dy) &= a\,dx \wedge (W_x\,dx + W_y\,dy) + a\,dy \wedge (W_x\,dx + W_y\,dy) \\
&= aW_y\,dx \wedge dy + bW_x\,dx \wedge .
\end{aligned}
$$

$H^0(K) = 0$ since W_x and W_y are both nonzerodivisors making s_1 injective. The first cohomology group is $\ker s_2/\operatorname{Im} s_1 = \ker r_y/\operatorname{Im} r_x \oplus \ker r_x/\operatorname{Im} r_y$. From Proposition 6.4, we recall that the partial derivatives of the nondegenerate polynomial form a regular sequence which means that W_y is a nonzerodivisor in $R/(W_x)$, so $\ker r_y/\operatorname{Im} r_x = 0$. Likewise with $\ker r_x/\operatorname{Im} r_y$. Hence $H^1(K) = 0$. The image of the map s_2 is going to be sums of multiples of W_x and W_y which is the ideal generated by (W_x, W_y) in R. Hence the second cohomology is going to $R/(W_x, W_y)\,dx \wedge dy$. Once again we can extende $K(W_x\,dx, W_y\,dy)$ to the exact sequence

$$0 \longrightarrow R \xrightarrow{s_1} R\,dx \oplus R\,dy \xrightarrow{s_2} R\,dx \wedge dy \longrightarrow R/(W_x, W_y)\,dx \wedge dy \longrightarrow 0$$

To iterate the Koszul complex construction to more variables, we give a definition that describes $R\,dx \wedge dy$ in terms of $R\,dx \oplus R\,dy$.

Definition 6.6. Let $R = \mathbb{C}[x_1, x_2, \ldots, x_n]$. Let N be the free module $\bigoplus_{i=1}^{n} R\,dx_i$. Then the *exterior algebra* is defined as $\wedge N = R \oplus N \oplus (N \otimes N) \oplus \ldots$ modulo the relations $dx \otimes dy = -dy \otimes dx$ and $dx \otimes dx = 0$ for all dx.

Definition 6.7. We define the m^{th} *wedge product of* N (denoted by $\wedge^m N$) to be the degree m part of $\wedge N$.

Note that $\wedge^{n+k} N = 0$ for $k > 0$, and $\wedge^1 N = N$. Also, since $\wedge^k N = \bigoplus_{i_1 < i_2 < \ldots < i_k} R\,dx_{i_1} \wedge \ldots \wedge dx_{i_k}$, it contains $\binom{n}{k}$ copies of R.

So going back to $K(W_x\,dx, W_y\,dy)$, we get that $R\,dx \oplus R\,dy = N$ and $R\,dx \wedge dy = \wedge^2 N$ which gives

$$0 \longrightarrow R \xrightarrow{s_1} N \xrightarrow{s_2} \wedge^2 N \longrightarrow 0$$

The Koszul complex in n variables is $K(W_{x_1}\,dx_1, W_{x_2}\,dx_2, \ldots, W_{x_n}\,dx_n)$, and is denoted as $K(W)$. It is the following sequence

$$0 \longrightarrow R \xrightarrow{s_1} N \xrightarrow{s_2} \wedge^2 N \xrightarrow{s_3} \ldots \longrightarrow \wedge^{n-1} N \xrightarrow{s_n} \wedge^n N \longrightarrow 0$$

From [Eis95], we get that in the Koszul complex, $H^j(K(W)) = 0$ for $j < n$ and $H^n(K(W)) = \wedge^n N/(W_{x_1}\,dx_1, \ldots, W_{x_n}\,dx_n) = (R/(W_{x_1}, \ldots, W_{x_n}))\,dx_1 \wedge \ldots \wedge dx_n$ and so we get an exact sequence

$$0 \longrightarrow R \xrightarrow{s_1} N \xrightarrow{s_2} \ldots \longrightarrow \wedge^{n-1} N \xrightarrow{s_n} \wedge^n N \longrightarrow (R/(W_{x_1}, \ldots, W_{x_n}))\,dx_1 \wedge \ldots \wedge dx_n \longrightarrow 0$$

Remark. Here, it is useful to note that by definition, $\wedge^k N = \Omega^k$ and so the n^{th} cohomology (which is the last term) is $\mathcal{H}_1$ as defined in Definition 2.10. To find H_h, for other $h \in G$, we just need to replace W with $W|_{\operatorname{Fix} h}$ and let n be the dimension of $\operatorname{Fix} h$. For each admissible group G we get a natural G action on each term of the Koszul complex that is given by multiplication on each variable. Since $dW = W_{x_1}\, dx_1 + W_{x_2}\, dx_2 + \ldots + W_{x_n}\, dx_n$, the action of G on dW is trivial since G fixes the polynomial W. Hence the map dW is equivariant.

6.3 Poincaré Polynomial of $(R/(W_{x_1}, \ldots, W_{x_n}))\, dx_1 \wedge \ldots \wedge dx_n$

Here we apply the formula in Proposition 5.10.

6.3.1 2 Variables. We first look at the two-variable case and use the previous tools to compute the representation-valued Poincaré polynomial of $\Omega^2/(dW \wedge \Omega^1)$ in two variables.

$$\begin{tikzcd}
0 \arrow[r] & N_0 = R \arrow[r, "r_x"] \arrow[dr, "r_y"] & N_1 = R\, dx \arrow[rr] \arrow[dr, "r_y"] & & 0 \\
& 0 \arrow[r] & N_1' = R\, dy \arrow[r, "-r_x"] & N_2 = R\, dx \wedge dy \arrow[r] & 0
\end{tikzcd}$$

where $r_x(f) = f \wedge W_x\, dx$ and $r_y(f) = f \wedge W_y\, dy$.

We start from the final term N_2. This is a single copy of R. We let q_x, q_y be the weights of x and y and ρ_x, ρ_y be the representations of G on $\operatorname{Span}\{x\}$ and $\operatorname{Span}\{y\}$ respectively. If we consider all possible monomials of in R, we get the Hilbert series $(1 + \rho_x t^{q_x} + \rho_x^2 t^{2q_x} + \ldots)(1 + \rho_y t^{q_y} + \rho_y^2 t^{2q_y} + \ldots)$ which can be simplified as a geometric series to $\dfrac{1}{1 - \rho_x t^{q_x}} \dfrac{1}{1 - \rho_y t^{q_y}}$. For all the terms in N_2, there is an extra G-action based on the $dx \wedge dy$, and so the Hilbert series for N_2 is $\dfrac{\rho_x \rho_y}{(1 - \rho_x t^{q_x})(1 - \rho_y t^{q_y})}$.

Looking to N_1, we also have a single copy of R which gives us once again the denominator of $(1 - \rho_x t^{q_x})(1 - \rho_y t^{q_y})$. All the terms in N_1 have an extra G-action based on the dx and so the numerator has a ρ_x factor. Also, the map from N_1 to N_2 is multiplication by $W_y\, dy$ which has degree $d - q_y$ since taking the derivative of W loses a single power of y. So the Hilbert series for N_1 is $\dfrac{\rho_x t^{d-q_y}}{(1-\rho_x t^{q_x})(1-\rho_y t^{q_y})}$. Likewise, the Hilbert series for N_1' is $\dfrac{\rho_y t^{d-q_x}}{(1-\rho_x t^{q_x})(1-\rho_y t^{q_y})}$.

Now for N_0. There are no attached dx_i's, but now all the terms increase by the degree of W_xW_y from N_0 to N_2. After accounting for that, we get that the Hilbert series for N_0 is $\dfrac{t^{d-q_y}t^{d-q_x}}{(1-\rho_x t^{q_x})(1-\rho_y t^{q_y})}$.

If we include $H^2(K(W))$, we would get an exact sequence. Hence the Poincaré polynomial for $H^2(K(W))$ would be $P(N_2) - P(N_1) - P(N_1') + P(N_0)$.

$$\begin{aligned} P(N_2) - P(N_1) - P(N_1') + P(N_0) &= \frac{\rho_x\rho_y}{(1-\rho_x t^{q_x})(1-\rho_y t^{q_y})} - \frac{\rho_x t^{d-q_y}}{(1-\rho_x t^{q_x})(1-\rho_y t^{q_y})} \\ &\quad - \frac{\rho_y t^{d-q_x}}{(1-\rho_x t^{q_x})(1-\rho_y t^{q_y})} + \frac{t^{d-q_y}t^{d-q_x}}{(1-\rho_x t^{q_x})(1-\rho_y t^{q_y})} \\ &= \frac{(\rho_x - t^{d-q_x})(\rho_y - t^{d-q_y})}{(1-\rho_x t^{q_x})(1-\rho_y t^{q_y})}. \end{aligned}$$

This gives us an explicit formula for the representation-valued Poincaré polynomial in two variables.

6.3.2 n Variables.

We now compute the representation-valued Poincaré polynomial in more variables. The term $\wedge^i N$ contains $\binom{n}{i}$ copies of R. For R itself, let q_i be the weight of x_i and let ρ_i be the representation of the action of G on x_i. Considering all possible monomials in R, we get that the Hilbert series for R is $(1+\rho_1 t^{q_1} + \rho_1^2 t^{2q_1} + \ldots)(1+\rho_2 t^{q_2} + \rho_2^2 t^{2q_2} + \ldots)\ldots(1+\rho_n t^{q_n} + \rho_n^2 t^{2q_n} + \ldots)$. Since each factor is a geometric series, we can express $(1+\rho_i t^{q_i} + \rho_i^2 t^{2q_i} + \ldots)$ as $\dfrac{1}{1-\rho_i t^{q_i}}$.

Every copy of R in $\wedge^i N$ would have the same denominator in the Poincaré polynomial, but the numerator would depend on the dx_i's. For $R\, dx_{m_1} \wedge dx_{m_2} \wedge \ldots \wedge dx_{m_i}$, every term would have an extra action of $\rho_{m_1}\rho_{m_2}\ldots\rho_{m_i}$ due to the dx_i's. At the same time, if we consider the degree of each term, based on the degree of the Koszul complex, we get that every term has an extra degree of $(d-q_{m_1'}) + (d-q_{m_2'}) + \ldots + (d-q_{m_{n-i}'})$, where the m''s are correspond to the other variables that are not part of $\{x_{m_1}, \ldots, x_{m_i}\}$.

$\wedge^n N$ has numerator $\rho_1\rho_2\ldots\rho_n$.

$\wedge^{n-1}N$ has numerator $\sum_i \rho_1\rho_2\ldots\rho_{i-1}\rho_{i+1}\ldots\rho_n t^{d-q_i}$.

$\wedge^{n-k}N$ has numerator $\sum_{i_1<i_2<\ldots<i_k} \rho_{i_1}\ldots\rho_{i_k} t^{\beta_{i_1,\ldots,i_k}}$ where $\beta_{i_1,\ldots,i_k} = \sum_{j\notin\{i_1,\ldots,i_k\}} d-q_j$.

$\wedge^0 N$ has numerator $t^{\sum_i d-q_i}$.

Taking $P(\wedge^n N) - P(\wedge^{n-1}N) + \ldots + (-1)^n P(N)$ and simplifying it combinatorially, we get the product of $(\rho_1 - t^{d-q_1})\ldots(\rho_n - t^{d-q_n})$. Dividing by the denominator, we get

$$\prod_{i=1}^{n} \frac{\rho_i - t^{d-q_i}}{1-\rho_i t^{q_i}}.$$

Theorem 6.8. *For any quasihomogeneous polynomial W and admissible group G. The Poincaré polynomial for $\mathcal{H}_h$ is $\prod_{i=1}^{m} \frac{\rho_i - t^{d-q_i}}{1-\rho_i t^{q_i}}$, where $x_1,\ldots,x_m$ are the variables in* $\operatorname{Fix} h$, *$q_1,\ldots,q_m$ are the corresponding weights of each variable and $\rho_1\ldots\rho_m$ are the representations of the action of G on the span of each variable.*

Based on this theorem, Theorem 5.2 becomes a direct corollary.

Corollary 6.9. *Let W be a non-degenerate quasihomogeneous polynomial with weights $d, q_1, q_2, \ldots, q_n$. Then the Poincaré polynomial of the Milnor ring $\mathcal{Q}_W$ without representation is*

$$P(\mathcal{Q}_W) = \prod_{i=1}^{n} \frac{1-t^{d-q_i}}{1-t^{q_i}}.$$

Proof. Since we are going to lose all information regarding the group action, we can ignore all dx_i's. The Milnor ring is isomorphic to $\mathcal{H}_h$ where h is the trivial group element and thus we can use the representation of the trivial group, whose representation ring is isomorphic to the complex numbers. $\square$

7.1 Group-Weights Conjecture

The proof of the Group-Weights Conjecture follows almost directly from the description of the Poincaré polynomial in Theorem 6.8.

Corollary 7.1. *Let W_1 and W_2 be polynomials which have the same weights. Suppose $G \leq G_{W_1}^{max}$ and $G \leq G_{W_2}^{max}$, then $\mathcal{H}_{W_1,G} \cong \mathcal{H}_{W_2,G}$ as graded vector spaces.*

Proof. First, we recall that Definition 2.13 does not give us the graded vector space structure of the FJRW-Ring. Rather, the grading of the FJRW ring is based on W-degree.

By definition, $\mathcal{H}_{W_1,G} = \oplus_{h \in G} (\mathcal{H}_h)^G$ and $\mathcal{H}_{W_2,G} = \oplus_{h \in G} (\mathcal{H}'_h)^G$ (note that $\mathcal{H}'_h$ has a prime symbol because $\mathcal{H}_h$ depends on the choice of polynomial). For each $h \in G$, every element in $\mathcal{H}_h$ has the same degree, and so in order to prove isomorphism, all we have to show is that for each h, $\dim (\mathcal{H}_h)^G = \dim (\mathcal{H}'_h)^G$.

The dimension of $(\mathcal{H}_h)^G$ is the dimension of the G-invariant representation of the Poincaré polynomial for $\mathcal{H}_h$. Since the formula for the Poincaré polynomial depends only upon the weights and the group chosen, $\dim (\mathcal{H}_h)^G = \dim (\mathcal{H}'_h)^G$. □

Recall from Section 2.5 that FJRW rings are deformation invariant which means that the Frobenius manifold structure of the FJRW-theory is dependent only on the associated state space. Hence by showing that $\mathcal{H}_{W_1,G_1} \cong \mathcal{H}_{W_2,G_2}$ as graded vector spaces, we have shown that $\mathcal{H}_{W_1,G_1} \cong \mathcal{H}_{W_2,G_2}$ in the FJRW-theory.

Theorem (Group-Weights). *Let W_1 and W_2 be polynomials which have the same weights. Suppose $G \leq G_{W_1}^{max}$ and $G \leq G_{W_2}^{max}$, then $\mathcal{H}_{W_1,G} \cong \mathcal{H}_{W_2,G}$ as FJRW A-models.*

Note that this theorem is a corollary to Corollary 7.1.

7.2 Using the Poincaré Polynomial

This section gives some insight as to the process of deriving the G-invariant portion of the Poincaré polynomial in Corollary 6.9. Here we focus on the sector $\mathcal{H}_1$ of the state space.

7.2.1 $W = x^n$**.** Let a Fermat atomic type polynomial be $W = x^n$ and with group $\langle e^{2\pi i \frac{1}{n}} \rangle$. We have that the weight of x is 1 and the total weight of W is n. So the Poincaré polynomial of $\mathcal{H}_1$ is $\dfrac{\rho - t^{n-1}}{1-\rho t}$ where ρ is the obvious representation on Span$\{x\}$. Note that $\rho^n = 1$.

The numerator factors: $(1-\rho t)(\rho^{n-1}t^{n-2} + \rho^{n-2}t^{n-3} + \ldots + \rho^2 t + \rho)$. Hence we get the Poincaré polynomial of the trivial sector as $(\rho^{n-1}t^{n-2} + \rho^{n-2}t^{n-3} + \ldots + \rho^2 t + \rho)$, which has no G-invariant terms.

7.2.2 $W = x^2y + xy^4$**.** Let a loop type polynomial be $W = x^2y + xy^4$ with weights $q_x = 3$ and $q_y = 1$ as shown in Chapter 2. We let the group $G = \langle e^{2\pi i \frac{3}{7}}, e^{2\pi i \frac{1}{7}} \rangle \cong C_7$.

If we let the action of G on the variable y be represented by ρ, where $\rho^7 = 1$, then it follows that the action of G on x can be represented by ρ^3.

So the Poincaré polynomial of $\mathcal{H}_1$ is

$$\begin{aligned}
\frac{\rho^3 - t^4}{1-\rho^3 t^3}\frac{\rho - t^6}{1-\rho t} &= \frac{(1-\rho t)(\rho^6 t^3 + \rho^5 t^2 + \rho^4 t + \rho^3)}{1-\rho^3 t^3}\frac{(1-\rho^3 t^3)(\rho^4 t^3 + \rho)}{1-\rho t} \\
&= (\rho^6 t^3 + \rho^5 t^2 + \rho^4 t + \rho^3)(t^3 + \rho) \\
&= \rho^3 t^6 + \rho^2 t^5 + \rho t^4 + 2t^3 + \rho^6 t^2 + \rho^5 t + \rho^4.
\end{aligned}$$

The term $2t^3$ is G-invariant and corresponds to the generators $y^3\, dx \wedge dy$ and $x\, dx \wedge dy \in (\mathcal{H}_1)^G$, both of degree 3.

Now in general, we can find an expression for the Poincaré polynomial of a loop type polynomial.

Suppose that x and y are variables of W. Let x and y have weights q_x and q_y respectively and let W have total weight d.

The term $x^a y$ is in the loop polynomial and so $aq_x + q_y = d$ or $aq_x = d - qy$.

Two of the terms in the product of the Poincaré polynomial of W are $\dfrac{\rho - t^{d-q_x}}{1-\rho t^{q_x}}$ and $\dfrac{\psi - t^{d-q_y}}{1-\psi t^{q_y}}$, where ρ and ψ are representations of x and y respectively.

We can show that $1-\rho t^{q_x}$ divides $\psi - t^{d-q_y}$ by first substituting $d-q_y$ with aq_x. Next, we note that since G fixes $x^a y$, $\rho^a\psi = 1$ and thus $\psi = \rho^{-a}$.

$$\begin{aligned}
\psi - t^{d-q_y} &= \rho^{-a} - t^{aq_y} \\
&= (1-\rho t^{q_x})(\rho^{-1}t^{q_x(a-1)} + \rho^{-2}t^{q_x(a-2)} + \ldots + \rho^{-(a-1)}t^{q_x} + \rho^{-a})
\end{aligned}$$

So if $W_{\text{loop}} = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \ldots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}x_1$, with representations $\rho_1, \ldots, \rho_n$, then $1-\rho_i t^{q_i}$ divides $\rho_{i+1} - t^{d-q_{i+1}}$ and we get that the Poincaré polynomial of the trivial sector of W_{loop} is

$$\prod_{i=1}^{n}\left(\sum_{j=1}^{a_i}\rho_i^{-j}t^{q_i(a_i-j)}\right).$$

7.3 Conclusion

At shown in Section 7.1, we have proved the Group-Weights conjecture. However, in the process of proving this conjecture we also now have a Poincaré polynomial for $\mathcal{H}_h$ in terms of representations. This formula also works for finding the state space of the B-model. The argument for proving a similar Group-Weight conjecture for the B-model does not work since deformation invariance does not exist in the B-model.

As illustrated in Section 7.2, finding the G-invariance of the Poincaré polynomial with representation is not as easy as compared to the proof of Theorem 5.3 where we just let $t = 1$ in order to find the general dimension of the Milnor ring. The hope is that studying more examples will lead to a formula for the G-invariance of the Poincaré polynomial with representation.

Bibliography

[Arn74] V.I. Arnold, *Normal forms of functions in the neighbourhoods of degenerate critical points*, Russian Math. surveys29, N2 1149 (1974).

[Eis95] David Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Springer-Verlag, 1995.

[FJJS11] A. Francis, T. Jarvis, D. Johnson, and R. Suggs, *Landau-Ginzburg mirror symmetry for orbifolded frobenius algebras*, eprint arXiv:1111.2508 [math.AG] (2011).

[FJR12] Huijun Fan, Tyler J. Jarvis, and Yongbin Ruan, *The Witten equation, mirror symmetry and quantum singularity theory*, Annals of Mathematics (to appear), e-print arXiv:0712.4021 [math.AG] (2012).

[Fra12] Amanda Francis, *New computational technique in FJRW theory with applications to Landau Ginzburg mirror symmetry*, Ph.D. thesis, Brigham Young University, 2012.

[JL01] Gordon James and Martin W Liebeck, *Representations and characters of groups*, Cambridge University Press, 2001.

[Kra09] Marc Krawitz, *FJRW rings and Landau-Ginzburg mirror symmetry*, eprint arXiv:0906.0796 [math.AG] (2009).

[KS92] Maximilian Kreuzer and Harald Skarke, *On the classification of quasihomogeneous functions*, Comm. Math. Phys. 150, no. 1, 137-147 (1992).

[Pic06] Anne Pichereau, *Poisson (co)homology and isolated singularities*, Journal of Algebra **299** (2006), p. 747-777.

[Sta79] Richard P. Stanley, *Invariants of finite groups and their applications to combinatoris*, Bull. Amer. Math. Soc. (N.S.) **1** (1979), p. 475-511.

[Stu93] Bernd Sturmfels, *Algorithms in invariant theory*, Springer-Verlag/Wien, 1993, p. 37.

[Wal80a] C.T.C. Wall, *A note on symmetry of singularities*, Bull. London Math. Soc. **12** (1980), no. 3, p. 169-175.

[Wal80b] ______, *A second note on symmetry of singularities*, Bull. London Math. Soc. **12** (1980), no. 5, p. 347-354.